Chapter 1

Metricke prostory $\mathbb{R}^n$ - teorie

Definice 1.1. *Necht* $n \in \mathbb{N}$, $\boldsymbol{x} = [x_1, \dots, x_n]$, $\boldsymbol{y} = [y_1, \dots, y_n] \in \mathbb{R}^n$, $r_0 \geq 0$ *a* $A,\ B \subset \mathbb{R}^n$.

- $\boldsymbol{o} := [0, \dots, 0]$, *(pocatek)*
- $A^C := \mathbb{R} \setminus A$, *(complement mnoziny A)*
- $\rho_e(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, *(Eukleidovska metrika)*
- $\operatorname{dist}(A, B) := \inf\{\rho_e(\boldsymbol{x}^1, \boldsymbol{x}^2);\ \boldsymbol{x}^1 \in A \wedge \boldsymbol{x}^2 \in B\}$, $A, B \neq \emptyset$,
- $B(\boldsymbol{x}, r_0) := \{\boldsymbol{z} \in \mathbb{R}^n;\ \rho_e(\boldsymbol{x}, \boldsymbol{z}) < r_0\}$, *(koule o stredu* $\boldsymbol{x}$ *a polomeru* r_0*)*
- $\operatorname{int}(A) := \{\boldsymbol{z} \in \mathbb{R}^n;\ \exists r > 0 :\ B(\boldsymbol{z}, r) \subset A\}$, *(vnitrni body mnoziny A)*
- $\operatorname{bd}(A) := \{\boldsymbol{z} \in \mathbb{R}^n;\ \forall r > 0 : B(\boldsymbol{z}, r) \cap A \neq \emptyset \wedge B(\boldsymbol{z}, r) \cap A^C \neq \emptyset\}$, *(hranicni body mnoziny A, obcas znacime* $H(A)$ *misto* $\operatorname{bd}(A)$*)*
- $\overline{A} := A \cup \operatorname{bd}(A)$. *(uzaver mnoziny A)*
- *Mnozina A je otevrena jestlize* $A = \operatorname{int}(A)$.
- *Mnozina A je uzavrena jestlize* $A = \overline{A}$.
- *Posloupnost* $\{\boldsymbol{x}^j\}$ *prvku* $\mathbb{R}^n$ *konverguje k* $\boldsymbol{x}$ *jestlize* $\lim_{j \to \infty} \rho_e(\boldsymbol{x}^j, \boldsymbol{x}) = 0$.
- *Mnozina A je omezena, jestlize existuje* $r > 0$ *takove, ze* $A \subset B(\boldsymbol{o}, r)$.

Nejprve uvedeme vlastnosti vyse definovanych pojmu, ktere se dokazuji na prednasce.

Veta 1.2. *Necht* $n \in \mathbb{N}$, $\boldsymbol{x} = [x_1, \dots, x_n]$, $\boldsymbol{y} = [y_1, \dots, y_n] \in \mathbb{R}^n$, $r_0 > 0$, $\lambda \in \mathbb{R}$ *a* $A \subset \mathbb{R}^n$. *Pak plati:*

(V1) ρ_e *je translacne invariantni metrika, ktera je homogenni.* $(\rho_e(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}) = |\lambda|\rho_e(\boldsymbol{x}, \boldsymbol{y}))$

(V2) $\mathbb{R}^n, \emptyset$ *jsou otevrene. Sjednoceni otevrenych a konecny prunik otevrenych je otevrena.*

- $B(\boldsymbol{x}, r_0)$ *je otevrena mnozina.*
- *Posloupnost* $\{\boldsymbol{x}^j\}$ *prvku* $\mathbb{R}^n$ *konverguje k* $\boldsymbol{x}$ *prave tehdy kdyz*

$$\forall \varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall j \geq j_0 :\ \boldsymbol{x}^j \in B(\boldsymbol{x}, \varepsilon).$$

(V3) *Posloupnost $\{\boldsymbol{x}^j\}$ prvku $\mathbb{R}^n$ konverguje k $\boldsymbol{x}$ prave tehdy kdyz konverguje po souradnicich.*

(V4) *Mnozina A je uzavrena prave tehdy kdyz A^C je otevrena a to je prave tehdy kdyz nelze z mnoziny A vykonvergovat posloupnosti.*

(V5) *$\mathbb{R}^n, \emptyset$ jsou uzavrene. Prunik uzavrenych a konecne sjednoceni uzavrenych je uzavrena.*

- *$A \subset B \subset \mathbb{R}^n$ pak $\operatorname{int}(A) \subset \operatorname{int}(B)$ a $\overline{A} \subset \overline{B}$.*

(V7) *Mnozina A je omezena prave tehdy kdyz je omezena mnozina $\overline{A}$.*

Proof. Dokazeme tvrzeni, ktera nejsou cislovanymi vetami z prednasky. Nejprve ukazeme, ze $B(\boldsymbol{x}, r_0)$ je otevrena. Staci ukazat, ze pro libovolne $\boldsymbol{z} \in B(\boldsymbol{x}, r_0)$ plati, ze $B(\boldsymbol{z}, R) \subset B(\boldsymbol{x}, r)$, kde $R = r - \rho_e(\boldsymbol{x}, \boldsymbol{z})$. Protoze $\boldsymbol{z} \in B(\boldsymbol{x}, r_0)$, tak $R > 0$. Necht $\boldsymbol{y} \in B(\boldsymbol{z}, R)$. Pak

$$\rho_e(\boldsymbol{x}, \boldsymbol{y}) \leq \rho_e(\boldsymbol{x}, \boldsymbol{z}) + \rho_e(\boldsymbol{z}, \boldsymbol{y}) < \rho(\boldsymbol{x}, \boldsymbol{z}) + R = r.$$

Tedy $\boldsymbol{y} \in B(\boldsymbol{x}, r)$.

Dale

$$\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x} \Leftrightarrow \lim_{j\to\infty} \rho_e(\boldsymbol{x}^j, \boldsymbol{x}) = 0 \Leftrightarrow (\forall \varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall j \geq j_0 :\ \rho_e(\boldsymbol{x}^j, \boldsymbol{x}) < \varepsilon) \Leftrightarrow$$
$$\Leftrightarrow (\forall \varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall j \geq j_0 :\ \boldsymbol{x}^j \in B(\boldsymbol{x}, \varepsilon)).$$

Necht $A \subset B$ a $\boldsymbol{x} \in \operatorname{int}(A)$. Pak existuje $r > 0$, ze $B(\boldsymbol{x}, r) \subset A \subset B$ tedy $\boldsymbol{x} \in \operatorname{int}(B)$.

Necht $A \subset B$. Pak $\overline{A} \subset \overline{B}$ plyne z Vety 1.4(vii). Bud lze prejit k doplnkum a vyuzit prvni rovnost a predchozi tvrzeni a nebo lze vyuzit druhou rovnost a dokazat primo.

□

Dale bych rad zminil dve trivialni vlastnosti pojmu dist.

Poznamka 1.3. *Necht $A, B \neq \emptyset$. Pak $\operatorname{dist}(A, B) \geq 0$ a $\operatorname{dist}(A, B) = \operatorname{dist}(B, A)$. Prvni vlastnost plyne primo z definice a nezapornosti metriky, kdezto druha ze symetricnosti metriky.*

Nyni si uvedeme nektere dalsi vlastnosti vyse definovanych pojmu.

Veta 1.4. *Necht $n \in \mathbb{N}$ and $A \subset \mathbb{R}^n$. Pak plati:*

(i) $\operatorname{int}(A) \subset A \subset \overline{A}$.

(ii) $\operatorname{int}(A) \cap \operatorname{bd}(A) = \emptyset$.

(iii) $\operatorname{bd}(A) = \operatorname{bd}(A^C)$.

(iv) $A \subset \operatorname{int}(A) \cup \operatorname{bd}(A)$.

(v) $\overline{A} = \operatorname{int}(A) \cup \operatorname{bd}(A)$.

(vi) $\mathbb{R}^n$ *je disjunktnim sjednocenim mnozin* $\operatorname{int}(A)$, $\operatorname{int}(A^C)$ *a* $\operatorname{bd}(A)$.

(vii) $\overline{A} = (\operatorname{int}(A^C))^C = \{\boldsymbol{z} \in \mathbb{R}^n;\ \forall r > 0 :\ B(\boldsymbol{z}, r) \cap A \neq \emptyset\}$.

(viii) $\operatorname{int}(A) = (\overline{A^C})^C$.

(ix) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$. *($\operatorname{int}(A)$ je otevrena)*

(x) $\overline{\overline{A}} = \overline{A}$. *($\overline{A}$ je uzavrena)*

(xi) $\operatorname{int}(A) = \bigcup\{G \subset A;\ G$ *je otevrena*$\}$. *(Mnozina* $\operatorname{int}(A)$ *je nejvetsi otevrena podmnozina A)*

(xii) $\overline{A} = \bigcap\{F \supset A;\ F$ *uzavrena*$\}$. *($\overline{A}$ je nejmensi uzavrena nadmnozina A)*

(xiii) $\overline{A} = \{\boldsymbol{z} \in \mathbb{R}^n;\ \operatorname{dist}(\{\boldsymbol{z}\}, A) = 0\}$, $A \neq \emptyset$.

(xiv) Necht A je otevrena i uzavrena. Pak $A \in \{\emptyset, \mathbb{R}^n\}$.

(xv) $\overline{A} = \{\boldsymbol{z} \in \mathbb{R}^n;$ *existuje posloupnost* $\{\boldsymbol{x}^j\}$ *prvku* $A:\ \lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{z}\}$.

(xvi) $\operatorname{bd}(A) = \overline{\operatorname{bd}(A)}$ *(hranice je uzavrena)*

(xvii) Funkce $f(x,y) := \rho_e(x,y) : \mathbb{R}^{2n} \to \mathbb{R}$ *je spojita.*

(xviii) Necht $G \subset \mathbb{R}^n$ *je otevrena a* $G \cap \overline{A} \neq \emptyset$. *Pak* $G \cap A \neq \emptyset$.

Dukaz.

(i) $\boldsymbol{x} \in \operatorname{int}(A) \Rightarrow (\exists r > 0:\ B(\boldsymbol{x}, r) \subset A) \Rightarrow \boldsymbol{x} \in A$ a definice $\overline{A}$.

(ii) $\boldsymbol{x} \in \operatorname{int}(A) \Rightarrow (\exists r > 0:\ B(\boldsymbol{x}, r) \subset A) \Rightarrow (\exists r > 0:\ B(\boldsymbol{x}, r) \cap A^C = \emptyset) \Rightarrow \boldsymbol{x} \notin \operatorname{bd}(A)$.

(iii) Okamzite z definice $\operatorname{bd}(A)$.

(iv) $\boldsymbol{x} \in A \Rightarrow (\forall r > 0:\ B(\boldsymbol{x}, r) \cap A \neq \emptyset)$ a $\boldsymbol{x} \notin \operatorname{int}(A) \Rightarrow (\forall r > 0:\ B(\boldsymbol{x}, r) \not\subset A) \Rightarrow (\forall r > 0:\ B(\boldsymbol{x}, r) \cap A^c \neq \emptyset)$. Pak z definice $\operatorname{bd}(A)$ plyne, ze $A \setminus \operatorname{int}(A) \subset \operatorname{bd}(A)$.

(v) $\overline{A} = A \cup \operatorname{bd}(A) \overset{(i)}{\supset} \operatorname{int}(A) \cup \operatorname{bd}(A) \overset{(iv)}{\supset} A \cup \operatorname{bd}(A) = \overline{A}$.

(vi) Disjunktnost plyne z (i), (ii) a (iii).

$$\mathbb{R}^n = A \cup A^C \overset{(iv)}{\subset} \operatorname{int}(A) \cup \operatorname{bd}(A) \cup \operatorname{int}(A^C) \cup \operatorname{bd}(A^C) \overset{(iii)}{=} \operatorname{int}(A) \cup \operatorname{bd}(A) \cup \operatorname{int}(A^C).$$

(vii)

$$(\overline{A})^C \overset{(v)}{=} (\operatorname{int}(A) \cup \operatorname{bd}(A))^C \overset{(vi)}{=} \operatorname{int}(A^C).$$

$$\boldsymbol{x} \notin \operatorname{int}(A^C) \Leftrightarrow (\forall r > 0:\ B(\boldsymbol{x}, r) \not\subset A^C) \Leftrightarrow (\forall r > 0:\ B(\boldsymbol{x}, r) \cap A \neq \emptyset).$$

(viii)

$$(\operatorname{int}(A))^C \overset{(vi)}{=} \operatorname{int}(A^C) \cup \operatorname{bd}(A) \overset{(iii)}{=} \operatorname{int}(A^C) \cup \operatorname{bd}(A^C) \overset{(v)}{=} \overline{A^C}.$$

(ix) $\boldsymbol{x} \in \operatorname{int}(A) \to (\exists r > 0:\ B(\boldsymbol{x}, r) \subset A)$. Protoze $B(\boldsymbol{x}, r)$ je otevrena mame

$$B(\boldsymbol{x}, r) = \operatorname{int}(B(\boldsymbol{x}, r)) \overset{B(\boldsymbol{x},r)\subset A}{\subset} \operatorname{int}(A).$$

Tedy $\boldsymbol{x} \in \operatorname{int}(\operatorname{int}(A))$. Opacna inkluze okamzite plyne z (i).

(x) Plyne okamzite z (V4), (vii) a (ix) prechodem k doplnkum.

(xi) Z (V2) plyne, ze mnozina napravo je otevrena a z (ix) plyne, ze $\operatorname{int}(A)$ je otevrena. Staci tedy dokazat, ze pro kazdou $B \subset \mathbb{R}^n$ takovou, ze $\operatorname{int}(A) \subsetneq B \subset A$ plati, ze B neni otevrena. Necht $\boldsymbol{z} \in B \setminus \operatorname{int}(A)$. Protoze $\operatorname{int}(B) \subset \operatorname{int}(A)$ mame $z \notin \operatorname{int}(B)$ a B neni otevrena.

(xii) Z (V5) plyne, ze mnozina napravo je uzavrena.

$$\overline{A} \overset{(vii)}{=} (\operatorname{int}(A^C))^C \overset{(xi)}{=} \left(\bigcup\{G \subset A^C;\ G \text{ je otevrena}\}\right)^C$$
$$\overset{DeMorgan}{=} \bigcap\{G^C;\ G \subset A^C \wedge\ G \text{ je otevrena}\} \overset{G^C=F,\ (V4)}{=} \bigcap\{F \supset A;\ F \text{ uzavrena}\}.$$

(xiii) $\boldsymbol{x} \in \overline{A} \overset{(vii)}{\Leftrightarrow} \boldsymbol{x} \in \{\boldsymbol{z} \in \mathbb{R}^n;\ \forall r > 0 :\ \ B(\boldsymbol{z}, r) \cap A \neq \emptyset\} \Leftrightarrow (\forall r > 0 :$ $\operatorname{dist}(A, \{\boldsymbol{x}\}) < r \Leftrightarrow \operatorname{dist}(A, \{\boldsymbol{x}\}) = 0$. Posledni ekvivalence vyuziva faktu, ze A je neprazdna (plyne z $\boldsymbol{x} \in \overline{A}$) a tedy $\operatorname{dist}(A, \{\boldsymbol{x}\}) \geq 0$.

(xiv) Predpokladejme, ze $A \notin \{\emptyset, \mathbb{R}^n\}$. Ukazeme, ze $\operatorname{int}(A) \neq \overline{A}$, coz je podle (ii) a (v) ekvivalentni s $\operatorname{bd}(A) \neq \emptyset$. Protoze $A \notin \{\emptyset, \mathbb{R}^n\}$ muzeme nalezt $\boldsymbol{x} \in A$ a $\boldsymbol{y} \in A^C$. Vzhledem k translacni invariantnosti metriky budeme predpokladat, ze $\boldsymbol{y} = \boldsymbol{o}$. Definujme

$$M := \{t \in [0, 1];\ t\boldsymbol{x} \in A\}.$$

Protoze $1 \in A$ a A je zdola omezena existuje $t_0 := \inf(M) \in [0, 1]$. Ukazeme, ze $\boldsymbol{z} := t_0\boldsymbol{x} \in \operatorname{bd}(A)$. Necht $r > 0$ je libovolne.

Nyni ukazeme, ze $B(\boldsymbol{z}, r) \cap A \neq \emptyset$. Pokud $t_0 = 1$, tak $\boldsymbol{z} = \boldsymbol{x}$ a $\boldsymbol{x} \in B(\boldsymbol{z}, r) \cap A$. Pokud $t_0 < 1$, tak definujeme

$$t_1^* := t_0 + \frac{r}{2\rho_e(\boldsymbol{x}, \boldsymbol{o})}.$$

Protoze $t_1^* > t_0$, muzeme nalezt $t_1 \in M$ takove, ze $t_1 < t_1^*$. Z definice M plyne, ze $t_1\boldsymbol{x} \in A$. Z definice t_1^* plyne, ze

$$\rho_e(\boldsymbol{z}, t_1\boldsymbol{x}) = (t_1 - t_0)\rho_e(\boldsymbol{x}, \boldsymbol{o}) < (t_1^* - t_0)\rho_e(\boldsymbol{x}, \boldsymbol{o}) = \frac{r}{2}.$$

Tedy $t_1\boldsymbol{x} \in B(\boldsymbol{z}, r)$.

Nakonec ukazeme, ze $B(\boldsymbol{z}, r) \cap A^C \neq \emptyset$. Pokud $t_0 = 0$, tak $\boldsymbol{z} = \boldsymbol{o}$ a $\boldsymbol{o} \in B(\boldsymbol{z}, r) \cap A^C$. Pokud $t_0 > 0$, tak definujeme

$$t_2 := \max\left\{0, t_0 - \frac{r}{2\rho_e(\boldsymbol{x}, \boldsymbol{o})}\right\}.$$

Protoze $t_2 < t_0$ mame $t_2 \notin M$. Protozea $t_2 \in [0, 1] \setminus M$ mame $t_2\boldsymbol{x} \in A^C$.

$$\rho_e(\boldsymbol{z}, t_2\boldsymbol{x}) = (t_0 - t_2)\rho_e(\boldsymbol{x}, \boldsymbol{o}) \leq \frac{r}{2}.$$

Tedy $t_2\boldsymbol{x} \in B(\boldsymbol{z}, r)$.

(xv) V nasledujicim dukazu opet pouzijeme nezapornost distance neprazdnych mnozin.

$$\boldsymbol{x} \in \overline{A} \overset{(xiii)}{\Rightarrow} \operatorname{dist}(A, \{x\}) = 0 \Rightarrow \left(\forall j \in \mathbb{N}\ \exists \boldsymbol{x}^j \in A :\ \rho(x^j, \boldsymbol{x}) < \frac{1}{j}\right) \Rightarrow$$
$$\Rightarrow (\exists \boldsymbol{x}^j \in A,\ j \in \mathbb{N} :\ \lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}) \Rightarrow$$
$$\Rightarrow (\exists a_j \in \mathbb{R},\ j \in \mathbb{N} :\ (\lim_{j\to\infty} a_j = 0 \wedge\ (\forall j \in \mathbb{N} :\ \operatorname{dist}(A, \{x\}) \leq a_j))) \Rightarrow$$
$$\Rightarrow \operatorname{dist}(A, \{x\}) = 0 \overset{(xiii)}{\Rightarrow} \boldsymbol{x} \in \overline{A}.$$

(xvi) Z (vi) plyne $\operatorname{bd}(A) = (\operatorname{int}(A) \cup \operatorname{int}(A^C))^C$. Zbytek plyne z (ix), (V2) a (V4).

(xvii) Z definice metriky plyne, ze

$$\rho_e([\boldsymbol{x}_0, \boldsymbol{y}_0], [\boldsymbol{x}_1, \boldsymbol{y}_1]) \geq \max\{\rho_e(\boldsymbol{x}_0, \boldsymbol{x}_1), \rho_e(\boldsymbol{y}_0, \boldsymbol{y}_1)\}.$$

Pak dvojim pouzitim trojuhelnikove nerovnosti obdrzime

$$\begin{aligned} |f(\boldsymbol{x}_0, \boldsymbol{y}_0) - f(\boldsymbol{x}_1, \boldsymbol{y}_1)| &= |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_0) - \rho_e(\boldsymbol{x}_1, \boldsymbol{y}_1)| \\ &\leq |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_0) - \rho_e(\boldsymbol{x}_0, \boldsymbol{y}_1)| + |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_1) - \rho_e(\boldsymbol{x}_1, \boldsymbol{y}_1)| \\ &\leq \rho_e(\boldsymbol{y}_0, \boldsymbol{y}_1) + \rho_e(\boldsymbol{x}_0, \boldsymbol{x}_1) \leq 2\rho_e([\boldsymbol{x}_0, \boldsymbol{y}_0], [\boldsymbol{x}_1, \boldsymbol{y}_1]), \end{aligned}$$

z cehoz plyne, ze f je spojita.

(xviii) Sporem: Necht $G \cap A = \emptyset$, pak existuje $\boldsymbol{x} \in G \cap \mathrm{bd}(A)$ a $\delta > 0$, ze $B(\boldsymbol{x}, \delta) \subset G$. Z definice $\mathrm{bd}(A)$ plyne, ze $\emptyset \neq B(\boldsymbol{x}, \delta) \cap A \subset G \cap A$, coz je spor.

$\square$

Poznamka 1.5. *Necht $A = \{\boldsymbol{x}^j;\ j \in \mathbb{N}\} \subset \mathbb{R}^n$. Pak A neni otevrena. (Otevrena neprazdna podmnozina $\mathbb{R}^n$ je nespocetna)*

Tato poznamka plyne ze znameho faktu, ze prvky intervalu $[0, 1]$ nelze usporadat do posloupnosti. Obdobne to lze dokazat pro libovolny nedegenerovany interval resp. usecku. Zbytek plyne z toho, ze kazda neprazdna otevrena podmnozina obsahuje neprazdnou otevrenou kouli a ta obsahuje nedegenerovanou usecku.

Chapter 2

Metricke prostory $\mathbb{R}^n$ - priklady

V nasledujicich prikladech obdrzime mnozinu A a pote podrobne urcime $\mathrm{int}(A)$, $\mathrm{bd}(A)$, $\overline{A}$ a to, zda mnozina A je otevrena, uzavrena ci ani jedno.

Priklad 2.1. *Necht $n \in \mathbb{N}$.*

(a) *Necht $a, b \in \mathbb{R}$. Definujme $A := (a, b >$. Pak $\mathrm{int}(A) = (a, b)$, $\mathrm{bd}(A) = \{a, b\}$, $\overline{A} =< a, b >$ (definice+V 1.4(vi)), mnozina A neni ani otevrena ($A \neq \mathrm{int}(A)$) ani uzavrena ($A \neq \overline{A}$)). Dusledkem tohoto prikladu je, ze kazdy otevreny interval je otevreny (V 1.4(ix)) a uzavreny interval je uzavreny (V 1.4(x)) (pro omezene intervaly). (Pro neomezene) Z V 1.2(V2) lze snadno odvodit, ze intervaly $(-\infty, a) = \bigcup_{n=1}^{\infty}(a - n, a)$ a $(a, +\infty) = \bigcup_{n=1}^{\infty}(a, a + n)$ jsou take otevrene. Z V 1.2(V4) plyne, ze $(-\infty, a >= (a, +\infty)^C$ a $< a, +\infty) = (-\infty, a)^C$ jsou uzavrene.*

(b) *$A := \{\boldsymbol{x} \in \mathbb{R}^n;\ \rho_e(\boldsymbol{x}, [0, \ldots, 0]) \in (0, 1 >\}$. Pak*

$$\begin{aligned} \mathrm{int}(A) =& \{\boldsymbol{x} \in \mathbb{R}^n;\ \rho_e(\boldsymbol{x}, \boldsymbol{o}) \in (0, 1)\}, \\ \mathrm{bd}(A) =& \{\boldsymbol{x} \in \mathbb{R}^n;\ \rho_e(\boldsymbol{x}, \boldsymbol{o}) = 1\} \cup \{\boldsymbol{o}\}, \ \textit{(definice+V 1.4(vi))} \\ \overline{A} =& \{\boldsymbol{x} \in \mathbb{R}^n;\ \rho_e(\boldsymbol{x}, \boldsymbol{o}) \leq 1\}. \end{aligned}$$

Mnozina A neni ani otevrena ($A \neq \mathrm{int}(A)$) ani uzavrena ($A \neq \overline{A}$)).

(c) *$A := \mathbb{Q}$. Pak $\mathrm{bd}(A) = \mathbb{R}^n$ (hustota racionalnich a iracionalnich cisel), $\mathrm{int}(A) = \emptyset$ (V 1.4(ii)), $\overline{A} = \mathbb{R}^n$ (definice). Mnozina A neni ani otevrena ($A \neq \mathrm{int}(A)$) ani uzavrena ($A \neq \overline{A}$)).*

(d) *$A := \mathbb{N}$. Pak $\mathrm{int}(A) \subset \mathrm{int}(\mathbb{Q}) = \emptyset$ (tez P 1.5). $\mathbb{N}^C = (-\infty, 1) \bigcup_{n=1}^{\infty}(n, n + 1)$ je otevrena ((a) a V 1.2(V2)). Tedy A je uzavrena a neni otevrena a $\overline{A} = \mathbb{N}$. $\mathbb{N} = \mathrm{bd}(A)$ ((V 1.4(v)).*

(e1) *$A := \{1/n;\ n \in \mathbb{N}\}$. Pak $\mathrm{int}(A) \subset \mathrm{int}(\mathbb{Q}) = \emptyset$ (tez P 1.5). $\mathrm{bd}(A) = \overline{A}$ ((V 1.4(v)). $\overline{A} = A \cup \{0\}$. $\overline{A} \supset A \cup \{0\}$ (V 1.4(xv)), $A \cup \{0\}$ je uzavrena (prez doplnek a uzavrenost otevrenych na sjednoceni) a V 1.4(xii).*

(e2) *Necht $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^j \in \mathbb{R}^n$, $j \in \mathbb{N}$, ze $\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$. Definujeme $A := \{\boldsymbol{x}^j;\ j \in \mathbb{N}\}$. Pak $\mathrm{int}(A) = \emptyset$ (P 1.5). $\mathrm{bd}(A) = \overline{A}$ ((V 1.4(v)). $\overline{A} = A \cup \{\boldsymbol{x}\}$ (V 1.4(xv)). Necht $\boldsymbol{y} \in \mathbb{R}^n \setminus (A \cup \{\boldsymbol{x}\})$. Polozime $R := \rho_e(\boldsymbol{x}, \boldsymbol{y})$ a zvolime $r \in (0, R)$. Pak $\exists j_0 \in \mathbb{N} :\ \forall j \geq j_0 :\ \boldsymbol{x}^j \in B(\boldsymbol{x}, r)$. Polozime $\delta := \min(\{R - r\} \cup \{\rho_e(\boldsymbol{y}, \boldsymbol{x}^j);\ j < j_0\})$. Pak $B(\boldsymbol{y}, \delta) \cap A = \emptyset$. Tedy $y \in \mathrm{int}(A^C)$ a $\boldsymbol{y} \notin \mathrm{bd}(A)$.*

(e3) *Necht* $a_n \in \mathbb{R}^n$, $n \in \mathbb{N}$. *Definujeme* $A := \{a_n;\ n \in \mathbb{N}\}$. *Pak* $\mathrm{int}(A) = \emptyset$ *(P 1.5).* $\mathrm{bd}(A) = \overline{A}$ *((V 1.4(v)).* $\overline{A}$ *viz V 1.4(xv))*

(f) $A := (-\infty, 0) \cup (\mathbb{Q} \cap (0, +\infty))$. $\mathrm{int}(A) \supset \mathrm{int}((-\infty, 0)) = (-\infty, 0)$, $\mathrm{bd}(A) \supset\ < 0, +\infty)$ *(hustota racionalnich a iracionalnich cisel). Z (V 1.4(ii))* $\mathrm{int}(A) = (-\infty, 0)$ *a* $\mathrm{bd}(A) =< 0, +\infty)$. *Z (V 1.4(V))* $\overline{A} = \mathbb{R}$. *Mnozina A neni ani otevrena* ($A \neq \mathrm{int}(A)$) *ani uzavrena* ($A \neq \overline{A}$)*.*

Priklad 2.2. *Necht* $n \in \mathbb{N}$ *a* $A, B \subset \mathbb{R}^n$. *Plati nasledujici tvrzeni?*

(α) $\overline{A} = \overline{\mathrm{int}(A)}$. *Ne* $A := \{0\}$. *Pouze* $\supset$.

(β) $\mathrm{int}(A) = \mathrm{int}(\overline{A})$. *Ne* $A := \{0\}^C$. $\subset$.

(γ) $\overline{A} = \overline{\mathrm{int}(\overline{A})}$ *Ne* $A := \{0\}$. $\supset$.

(δ) $\mathrm{int}(A) = \mathrm{int}(\overline{\mathrm{int}(A)})$. *Ne* $A := \{0\}^C$. $\subset$.

(ϵ) $\overline{\mathrm{int}(A)} = \overline{\mathrm{int}(\overline{A})}$. *Ne* $A := \mathbb{Q}$. $\subset$.

(μ) $\overline{\mathrm{int}(A)} = \overline{\mathrm{int}(\overline{\mathrm{int}(A)})}$. *Ano. Pouzivame V 1.4(i), (ix), (x), (xi), (xii). Pak* $\mathrm{int}(\overline{\mathrm{int}(A)}) \supset \mathrm{int}(A)$ ($\mathrm{int}(\overline{\mathrm{int}(A)})$ *je nejvetsi otevrena podmnozina* $\overline{\mathrm{int}(A)}$) *a tedy* $\overline{\mathrm{int}(\overline{\mathrm{int}(A)})} \supset \overline{\mathrm{int}(A)}$. *Na druhou stranu* $\overline{\mathrm{int}(A)}$ *je uzavrena nadmnozina* $\mathrm{int}(\overline{\mathrm{int}(A)})$ *a tedy* $\overline{\mathrm{int}(\overline{\mathrm{int}(A)})} \subset \overline{\mathrm{int}(A)}$.

(ψ) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. *Ano. Plyne okamzite z V 1.4(xiii) nebo (xv).*

(ϕ) $\overline{A \cap B} = \overline{A} \cap \overline{B}$. *Ne* $A :=< 0, 1)$ *a* $B := (1, 2 >$.

Priklad 2.3. *Pro danou mnozinu A nacrtnete A,* $\mathrm{int}(A)$, $\overline{A}$, $\mathrm{bd}(A)$ *a rozhodnete zda je mnozina A otevrena, uzavrena ci ani jedno.*

(I) $A := B(\boldsymbol{o}, 1)$. *(Jednotkova koule.)*

(II) $A := \{[x, y] \in \mathbb{R}^2;\ x > 0, y \leq 0\}$. *(4. kvadrant bez osy y.)*

(III) $A := \{[x, y] \in \mathbb{R}^2;\ x^2 + e^y > 17\}$. ($y > \log(17 - x^2)$, $x \in (-\sqrt{17}, \sqrt{17})$ *nebo* $y \in \mathbb{R}$, $x \notin (-\sqrt{17}, \sqrt{17})$*).*

(IV) $A := \{[x, y] \in \mathbb{R}^2;\ x^2 + y^2 + 2xy = 5\}$. ($x + y \in \{\pm\sqrt{5}\}$)

(V) $A := \{[x, y, z] \in \mathbb{R}^3;\ x \geq 0,\ y > 0,\ x + y = 2,\ z \leq 0\}$.

(VI) $A := \{[x, y] \in \mathbb{R}^2;\ |x + y| - x - y > 0\}$. ($x + y < 0$)

(VII) $A := \{[x, y] \in \mathbb{R}^2;\ (x^2 + y^2)(1 - x^2 - y^2) \leq 0\}$. ($(B(0, 1))^C \cup \{[0, 0]\}$)

(VIII) $A := \left\{[x, y] \in \mathbb{R}^2;\ \frac{4 - 4x^2 - y^2}{4y} \geq 0\right\}$. *Mejme elipsu* $4x^2 + y^2 = 4$. *Pak mnozina A je v 1. a 2. kvadrantu uvnitr teto elipsy a v 3. a 4. kvadrantu vne. Obvod elipsy do mnoziny patri, osa y take, ale osa x ne (vcetne pruniku s osou y a elipsou).*